Completeness of Bethe ansatz for 1D Hubbard model with AB-flux through combinatorial formulas and exact enumeration of eigenstates

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Abstract

For the one-dimensional Hubbard model with Aharonov-Bohm-type magnetic flux, we study the relation between its symmetry and the number of Bethe states. First we show the existence of solutions for Lieb-Wu equations with an arbitrary number of up-spins and one down-spin, and exactly count the number of the Bethe states. The results are consistent with Takahashi's string hypothesis if the system has the $so(4)$ symmetry. With the Aharonov-Bohm-type magnetic flux, however, the number of Bethe states increases and the standard string hypothesis does not hold. In fact, the $so(4)$ symmetry reduces to the direct sum of charge-$u(1)$ and spin-$sl(2)$ symmetry through the change of AB-flux strength. Next, extending Kirillov's approach [12, 13], we derive two combinatorial formulas from the relation among the characters of $so(4)$- or $\big(u(1) \oplus sl(2)\big)$-modules. One formula reproduces Essler-Korepin-Schoutens' combinatorial formula for counting the number of Bethe states in the $so(4)$-case. From the exact analysis of the Lieb-Wu equations, we find that another formula corresponds to the spin-$sl(2)$ case.

1 Introduction

Low-dimensional physics has attracted a great interest of theoretical and experimental physicists for almost a half century [16, 22, 23]. Among them, the Bethe ansatz method, which was originally developed as a non-perturbative method for diagonalizing one-dimensional spin-$\frac{1}{2}$ isotropic Heisenberg spin chain [1], opened a new realm of mathematical physics. Roughly

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speaking, Bethe's work consists of three parts: i) Bethe states are introduced and Bethe equations, which are sufficient conditions for the Bethe states being eigenstates, are derived; ii) the existence of solutions for the Bethe equations are discussed in simple cases, and the general forms of solutions are conjectured, which are called string hypothesis; iii) under the string hypothesis, the formula for counting the number of the Bethe states is provided, which leads to the combinatorial completeness of Bethe ansatz. At present the Bethe ansatz method is applied to various kinds of one-dimensional spin chains and strongly correlated electron systems [16, 22, 30].

In Bethe's work, the combinatorial formula for counting the number of Bethe states possesses a wealth of mathematical implications. In the case of the isotropic Heisenberg spin chain, the Bethe states constructed by finite-valued solutions of Bethe equations do not produce all the eigenstates. In fact the system has $sl(2)$ symmetry and the Bethe states are $sl(2)$-highest [6]. The eigenstates other than highest weight vectors, i.e., $sl(2)$-descendant states, are constructed by applying the lowering operator to the Bethe states. Mathematically, the number of Bethe states is interpreted as the multiplicity of irreducible components in the tensor products of two-dimensional highest weight $sl(2)$-modules. Bethe's formula is also extended to the generalized Heisenberg spin chains with higher spins or $sl(n)$ symmetry, for which the powerful tools such as Q-systems are introduced [12, 13].

The application of Bethe ansatz method to the one-dimensional Hubbard model was given by Lieb and Wu [21]. The Bethe equations for the Hubbard model are often called Lieb-Wu equations. Takahashi's string hypothesis asserts that, in the thermodynamic limit, the solutions of Lieb-Wu equations are approximated by string solutions, and the number of Bethe states is estimated under the hypothesis [28, 29]. The Hubbard model with even sites has $so(4)$ symmetry [7, 32, 33]. Essler, Korepin and Schoutens proved that, when the system has the $so(4)$ symmetry, the Bethe states are $so(4)$-highest [5]. They also showed in a combinatorial way that all the eigenstates are obtained by taking the $so(4)$-descendants of the Bethe states into account [3]. On the other hand the completeness of Bethe ansatz for the system with odd sites, which has just $sl(2)$ symmetry related to the spin degrees of freedom, has not been discussed.

In this article, we study the Bethe states in the one-dimensional Hubbard model. In particular, we deal with the system on a ring with Aharonov-Bohm-type magnetic flux. The system has $so(4)$ symmetry only at special values of the AB-flux strength and the $so(4)$ symmetry reduces to spin-$sl(2)$ symmetry for other values. More precisely, for a generic value of the AB-flux strength, the $so(4)$ symmetry breaks into the direct sum of the charge-$u(1)$ and the spin-$sl(2)$ symmetry. Varying the AB-flux strength, we investigate solutions

of Lieb-Wu equations. Here we recall that all the enumeration of Bethe states we have mentioned above are based on the string hypothesis. However, the violation of the string hypothesis is numerically observed: for the spin-$\frac{1}{2}$ isotropic Heisenberg spin chain, some of the string solutions reduce to real solutions when the number of sites is large, which is called redistribution phenomenon [4, 8, 9]. Thus, without making any approximation, we discuss the existence of Bethe ansatz solutions with the AB-flux. In particular, we show the existence of solutions of the Lieb-Wu equations with an arbitrary number of up-spins and one down-spin. Here we employ a graphical approach [2, 24]. We exactly count the number of solutions in the case and verify that the enumeration with the string hypothesis is correct only in the $so(4)$-case. We find that the Lieb-Wu equations for the system with only the spin-$sl(2)$ symmetry have more solutions than those in the $so(4)$-case. Next we study the combinatorial completeness of Bethe ansatz. We obtain the relation among the characters of $so(4)$-modules through the power series identities similar to Kirillov's [12, 13], which gives a new proof of Essler-Korepin-Schoutens' combinatorial completeness of Bethe ansatz [3]. We also introduce a new combinatorial formula derived from the relation among the characters of $\big(u(1) \oplus sl(2)\big)$-modules. The formula suggests the combinatorial completeness of Bethe ansatz for the system only with the spin-$sl(2)$ symmetry, which has not been discussed in the literature.

The Bethe ansatz solutions with the AB-flux should be quite important in the low-dimensional physics of the one-dimensional Hubbard model. As pointed out by Kohn, the low frequency conductivity is directly related to the shift of the energy levels due to twisted boundary conditions [15]. Sharpening Kohn's argument on electron systems in any dimensions, Shastry and Sutherland discussed effective mass of the one-dimensional Hubbard model through the twisted boundary conditions [25]. Furthermore, Kawakami and Yang obtained an explicit expression for the effective mass of the electric conductivity for the one-dimensional Hubbard model [10, 11]. For the Bethe ansatz solutions with the twisted boundary conditions, there are other aspects such as persistent current associated with the AB-flux.

The article is organized as follows: in Section 2, we review the symmetry of the one-dimensional Hubbard model and the Bethe ansatz method. We also describe how to construct eigenstates other than the Bethe states. In Section 3 we prove the existence of solutions for the Lieb-Wu equations with one down-spin and exactly count the number of Bethe states in varying the strength of AB-flux. In Section 4 we study the combinatorial formulas for counting the Bethe states in terms of the characters of $so(4)$- and $\big(u(1) \oplus sl(2)\big)$-modules. The final section is devoted to summary and concluding remarks.

2 Bethe ansatz method

2.1 Hubbard model and Lieb-Wu equations

We introduce the Hubbard model on an L-site ring with Aharonov-Bohm-type magnetic flux Φ. Let $c_{is}^{\dagger}$ and $c_{is}, (i \in \mathbb{Z}/L\mathbb{Z}, s \in \{\uparrow, \downarrow\})$ be the creation and annihilation operators of electrons satisfying $\{c_{is}, c_{jt}\} = \{c_{is}^{\dagger}, c_{jt}^{\dagger}\} = 0$ and $\{c_{is}, c_{jt}^{\dagger}\} = \delta_{ij}\delta_{st}$, and define the number operators by $n_{is} := c_{is}^{\dagger}c_{is}$. We consider the Fock space V of electrons with the vacuum state $|0\rangle$ $(\dim V = 4^L)$. The one-dimensional Hubbard model is described by the following Hamiltonian acting on V:

$$H_{\phi} = -\sum_{1 \leqslant i \leqslant L} \sum_{s=\uparrow,\downarrow} (\mathrm{e}^{\sqrt{-1}\phi} c_{is}^{\dagger} c_{i+1,s} + \mathrm{e}^{-\sqrt{-1}\phi} c_{i+1,s}^{\dagger} c_{is}) + U \sum_{1 \leqslant i \leqslant L} \Big(n_{i\uparrow} - \frac{1}{2}\Big)\Big(n_{i\downarrow} - \frac{1}{2}\Big), \quad (2.1)$$

where we assume $\phi := \Phi/L \in \mathbb{R}/2\pi\mathbb{R}$ and $U > 0$. It is clear that $H_{\phi} = H_{\phi+2\pi}$. Furthermore H_{ϕ} has the same energy spectra as those of $H_{\phi+\frac{2\pi}{L}}$. Hence we often restrict the region of ϕ to $0 \leqslant \phi < \frac{2\pi}{L}$ in what follows.

Let N be the number of electrons and M that of down-spins. We assume $0 \leqslant 2M \leqslant N \leqslant L$. Let $\{k_i | i = 1, 2, \ldots, N\}, (\mathrm{Re}(k_i) \in \mathbb{R}/2\pi\mathbb{R})$ denote a set of wavenumbers of N electrons and $\{\lambda_{\alpha} | \alpha = 1, 2, \ldots, M\}$ that of rapidities of M down-spins. Given a set of spin configuration $\{s_i | i = 1, 2, \ldots, N\}$ with $N - M$ up-spins and M down-spins, the Bethe state with $\{k_i, \lambda_{\alpha}\}$ has the following form:

$$|k, \lambda; s\rangle_{N,M}^{\phi} = \sum_{\{1 \leqslant x_i \leqslant L\}} \psi_{k,\lambda}(x; s) c_{x_1,s_1}^{\dagger} c_{x_2,s_2}^{\dagger} \cdots c_{x_N,s_N}^{\dagger} |0\rangle. \quad (2.2)$$

The coefficients $\psi_{k,\lambda}(x; s)$ in (2.2) are explicitly given in [31]. The Bethe states (2.2) are eigenstates of the Hamiltonian (2.1) if $\{k_i, \lambda_{\alpha}\}$ satisfy the following equations:

$$\begin{aligned} \mathrm{e}^{\sqrt{-1}k_i L} &= \prod_{1 \leqslant \beta \leqslant M} \frac{\lambda_{\beta} - \sin(k_i + \phi) - \sqrt{-1}U/4}{\lambda_{\beta} - \sin(k_i + \phi) + \sqrt{-1}U/4}, \\ \prod_{1 \leqslant i \leqslant N} \frac{\lambda_{\alpha} - \sin(k_i + \phi) - \sqrt{-1}U/4}{\lambda_{\alpha} - \sin(k_i + \phi) + \sqrt{-1}U/4} &= \prod_{\beta(\neq \alpha)} \frac{\lambda_{\alpha} - \lambda_{\beta} - \sqrt{-1}U/2}{\lambda_{\alpha} - \lambda_{\beta} + \sqrt{-1}U/2}, \end{aligned} \quad (2.3)$$

which are coupled nonlinear equations called Lieb-Wu equations [21]. The Lieb-Wu equations have not been solved analytically. However it predicts some important results on thermodynamic properties of the system through Takahashi's string hypothesis [16, 28–30]. In terms of the solutions $\{k_i, \lambda_{\alpha}\}$ of the Lieb-Wu equations (2.3), energy eigenvalues are written as

$$H_{\phi}|k, \lambda; s\rangle_{N,M}^{\phi} = E|k, \lambda; s\rangle_{N,M}^{\phi}, \quad E = -2\sum_{1 \leqslant i \leqslant N} \cos(k_i + \phi) + \frac{1}{4}U(L - 2N). \quad (2.4)$$

Recall that the Bethe states (2.2) give only the eigenstates with $0 \leqslant 2M \leqslant N \leqslant L$. In order to construct other eigenstates, we need to consider the symmetries of the system.

Hereafter we shall sometimes abbreviate the superscript ϕ in $|k, \lambda; s\rangle^{\phi}_{N,M}$.

2.2 Symmetries

The U-independent symmetries of the Hubbard model are classified in [7]. First we review the symmetries connected with the spin and charge degrees of freedom [26, 27, 32, 33]. Define the following operators related to the spin degrees of freedom:

$$S_z := \frac{1}{2} \sum_{1 \leqslant i \leqslant L} (n_{i\uparrow} - n_{i\downarrow}), \quad S_+ := \sum_{1 \leqslant i \leqslant L} c^{\dagger}_{i\uparrow} c_{i\downarrow}, \quad S_- := (S_+)^{\dagger}. \tag{2.5}$$

They give a representation of the algebra $sl(2)$ on the Fock space V. Since all the operators $\{S_z, S_{\pm}\}$ commute with H_ϕ (2.1) for an arbitrary value of ϕ, it is said that the system has spin-$sl(2)$ symmetry. One finds another representation of $sl(2)$ related to the charge degrees of freedom,

$$\eta_z := \frac{1}{2} \sum_{1 \leqslant i \leqslant L} (1 - n_{i\uparrow} - n_{i\downarrow}), \quad \eta_+ := \sum_{1 \leqslant i \leqslant L} \mathrm{e}^{\sqrt{-1}(2\phi+\pi)i} c_{i\downarrow} c_{i\uparrow}, \quad \eta_- := (\eta_+)^{\dagger}. \tag{2.6}$$

Note that all the operators in (2.6) are commutative with $\{S_z, S_{\pm}\}$ (2.5). It is easy to see that the operator η_z also commutes with H_ϕ for an arbitrary value of ϕ. However other operators $\eta_{\pm}$ commute with H_ϕ only for the special values of ϕ satisfying $\frac{L}{2} + \frac{L\phi}{\pi} \in \mathbb{Z}$. Thus the system has charge-$sl(2)$ symmetry if $\frac{L}{2} + \frac{L\phi}{\pi} \in \mathbb{Z}$, and it reduces to charge-$u(1)$ symmetry given by η_z for other values of ϕ. Combining the above two kinds of $sl(2)$ symmetries, we see that the system has $so(4)(\simeq sl(2) \oplus sl(2))$ symmetry if $\frac{L}{2} + \frac{L\phi}{\pi} \in \mathbb{Z}$, and $u(1) \oplus sl(2)$ symmetry otherwise. For simplicity, we call the former $so(4)$-case and the latter $sl(2)$-case. For the later discussion, we define the Casimir operators for each $sl(2)$,

$$\boldsymbol{\eta}^2 := \frac{1}{2}\Big(\eta_+\eta_- + \eta_-\eta_+\Big) + \eta_z^2, \quad \boldsymbol{S}^2 := \frac{1}{2}\Big(S_+S_- + S_-S_+\Big) + S_z^2,$$

which are employed to see the dimension of each representation.

Next we introduce the following three operators:

$$T_{\mathrm{s}} := \prod_{1 \leqslant i \leqslant L} P_{i\uparrow;\, i\downarrow}, \quad T_{\mathrm{ph}} := \prod_{1 \leqslant i \leqslant L} \prod_{s=\uparrow,\downarrow} (c^{\dagger}_{is} + c_{is}), \quad T_{\mathrm{r}} := \prod_{1 \leqslant i \leqslant \lfloor \frac{L-1}{2} \rfloor} \prod_{s=\uparrow,\downarrow} P_{is;\, L-i,s},$$

where $P_{is;\, jt} := 1 - (c^{\dagger}_{is} - c^{\dagger}_{jt})(c_{is} - c_{jt})$ and $\lfloor x \rfloor$ denotes the greatest integer in x. One notices that T_{s} induces a spin-reversal transformation, T_{ph} a particle-hole transformation and T_{r} a reflection of the lattice. Direct calculation shows

$$T_{\mathrm{s}}^{-1} H_\phi T_{\mathrm{s}} = H_\phi, \quad T_{\mathrm{ph}}^{-1} H_\phi T_{\mathrm{ph}} = H_{\pi-\phi}, \quad T_{\mathrm{r}}^{-1} H_\phi T_{\mathrm{r}} = H_{-\phi}.$$

For the system with even L, i.e., a bipartite lattice, we also introduce

$$T_{\rm b} := \prod_{1\leqslant i\leqslant \frac{L}{2}} \prod_{s=\uparrow,\downarrow} (-1)^{n_{2i,s}}, \quad T_{\rm b}^{-1} H_\phi T_{\rm b} = H_{\phi+\pi}.$$

Combining these operations, we obtain the following transformation properties of the Hamiltonian H_ϕ (2.1):

$$\begin{aligned} &T_{\rm s}^{-1} H_\phi T_{\rm s} = H_\phi, \qquad T_{\rm ph}^{-1} T_{\rm r}^{-1} H_\phi T_{\rm r} T_{\rm ph} = H_{\phi+\pi}, \quad \text{for even and odd } L, \\ &T_{\rm ph}^{-1} T_{\rm r}^{-1} T_{\rm b}^{-1} H_\phi T_{\rm b} T_{\rm r} T_{\rm ph} = H_\phi, \quad \text{for even } L. \end{aligned} \tag{2.7}$$

One notices that for both even and odd L, the system has spin-reversal symmetry. While the system with even L has particle-hole symmetry brought by the transformation $T_{\rm b}T_{\rm r}T_{\rm ph}$, the system with odd L does not have particle-hole symmetry for generic values of ϕ except the special values satisfying $\frac{L}{2} + \frac{L\phi}{\pi} \in \mathbb{Z}$

2.3 Construction of non-Bethe states

We construct eigenstates that are not included in the Bethe states (2.2). First we consider the system with $so(4)$ symmetry, i.e., both spin-$sl(2)$ and charge-$sl(2)$ symmetries. Here we recall that the AB-flux parameter ϕ takes a special value: $\frac{L}{2} + \frac{L\phi}{\pi} \in \mathbb{Z}$ for even or odd L. Then, as shown in [5] for even L with $\phi = 0$, the Bethe states (2.2) characterized by finite-valued solutions of the Lieb-Wu equations (2.3) are $so(4)$-highest, i.e., $\eta_+|k,\lambda;s\rangle_{N,M} = S_+|k,\lambda;s\rangle_{N,M} = 0$. Since

$$\boldsymbol{\eta}^2|k,\lambda;s\rangle_{N,M} = \eta(\eta+1)|k,\lambda;s\rangle_{N,M}, \quad \boldsymbol{S}^2|k,\lambda;s\rangle_{N,M} = S(S+1)|k,\lambda;s\rangle_{N,M}, \tag{2.8}$$

with $\eta = \frac{1}{2}(L–N)$ and $S = \frac{1}{2}(N–2M)$, the Bethe state $|k,\lambda;s\rangle_{N,M}$ is the highest weight vector of an $(L-N+1)(N-2M+1)$-dimensional highest weight $so(4)$-module. Here we note that $\eta + S$ is an integer for even L [33], while it is a half-integer for odd L. The $so(4)$-descendant states of the Bethe state $|k,\lambda;s\rangle_{N,M}$,

$$(\eta_-)^n (S_-)^m |k,\lambda;s\rangle_{N,M}, \quad (0 < n \leqslant L-N, 0 < m \leqslant N-2M), \tag{2.9}$$

are also eigenstates of H_ϕ (2.1) (see Fig. 1). They are energy degenerate with $|k,\lambda;s\rangle_{N,M}$. It is easy to see that such application of the lowering operators η_- and S_- to the Bethe states produces other eigenstates than those with $0 \leqslant 2M \leqslant N \leqslant L$. Essler, Korepin and Schoutens counted the number of Bethe states (2.2) and their $so(4)$-descendant states (2.9) under the string hypothesis to show the combinatorial completeness of Bethe ansatz [3].

Next we consider the system only with the charge-$u(1)$ and the spin-$sl(2)$ symmetries. The Bethe states (2.2) are $sl(2)$-highest and satisfy only the second relation in (2.8), which means

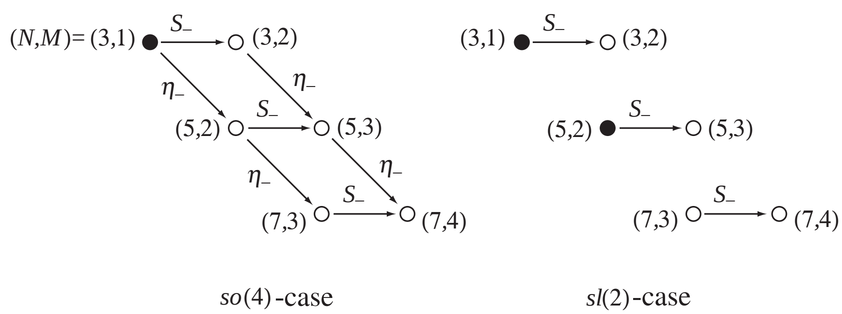

Figure 1: Bethe states and non-Bethe states for $L = 5$. The Bethe state with $(N, M) = (3, 1)$ denoted by a closed circle produces five $so(4)$-descendants denoted by open circles if the system has $so(4)$ symmetry. As the $so(4)$ symmetry reduces to $sl(2)$ symmetry, the six eigenstates form three doublets of $sl(2)$. Then we need one more Bethe state with $(5, 2)$. Note that the eigenstate with $(7, 3)$ is not a Bethe states and is constructed through the transformation $T_{\rm r}T_{\rm ph}$.

that the Bethe state $|k, \lambda; s\rangle^{\phi}_{N,M}$ is the highest weight vector of an $(N-2M+1)$-dimensional highest weight $sl(2)$-module. The $sl(2)$-descendant states of the Bethe states

$$(S_-)^m|k, \lambda; s\rangle^{\phi}_{N,M}, \quad (0 < m \leqslant N - 2M), \tag{2.10}$$

are eigenstates of H_ϕ (2.1) (see Fig. 1). One notices here that the Bethe states (2.2) and their $sl(2)$-descendants (2.10) do not produce the eigenstates with $L < N \leqslant 2L$ since the application of the lowering operator S_- does not change the number of electrons. Such eigenstates with $L < N \leqslant 2L$ are constructed as follows: i) by applying the transformation $T_{\rm b}T_{\rm r}T_{\rm ph}$ to the Bethe state $|k, \lambda; s\rangle^{\phi}_{2L-N,L-M}, (L < N \leqslant 2M \leqslant 2L)$ if L is even, or by applying $T_{\rm r}T_{\rm ph}$ to the Bethe state $|k, \lambda; s\rangle^{\phi+\pi}_{2L-N,L-M}, (L < N \leqslant 2M \leqslant 2L)$ obtained by the Lieb-Wu equations (2.3) with $\phi+\pi$ instead of ϕ if L is odd, we get the lowest weight vector of a highest weight $sl(2)$-module; ii) the application of the raising operators $(S_+)^n, (0 < n \leqslant 2M - N)$ to the lowest weight vector produces other degenerate eigenstates.

Even if we are interested only in the system with $\phi = 0$, the Bethe ansatz method needs the system with $\phi = \pi$ for odd L. One of the main purposes in this article is therefore to discuss whether or not the above procedure produces all the eigenstates.

3 Lieb-Wu equations with one down-spin

By employing a graphical approach, we exactly count the number of finite-valued solutions for the Lieb-Wu equations (2.3) in the case when the system contains an arbitrary number of up-spins and one down-spin, i.e., $N \geqslant 2$ and $M = 1$. We remark that such exact analysis is presented in [2, 24] for the case $\phi = 0$. In this section, we assume $0 \leqslant \phi < \frac{2\pi}{L}$ since H_ϕ has the

same energy spectra as those of $H_{\phi+\frac{2\pi}{L}}$. In the case $M=1$, the string hypothesis [28] predicts that two types of solutions exist; one is the solution with only real wavenumbers $\{k_i\}$ and another includes two complex wavenumbers. We investigate such types of solutions below.

3.1 Real k solutions

First we consider the real solutions. For $M=1$, the Lieb-Wu equations (2.3) reduce to

$$\begin{aligned} \mathrm{e}^{\sqrt{-1}k_iL} &= \frac{\lambda-\sin(k_i{+}\phi)-\sqrt{-1}U/4}{\lambda-\sin(k_i{+}\phi)+\sqrt{-1}U/4}, \quad (i=1,2,\dots,N), \\ \prod_{1\leqslant i\leqslant N} & \frac{\lambda-\sin(k_i{+}\phi)-\sqrt{-1}U/4}{\lambda-\sin(k_i{+}\phi)+\sqrt{-1}U/4} = 1. \end{aligned} \tag{3.1}$$

These are equivalent to the following equations:

$$\sin(k_i{+}\phi)-\lambda = \frac{U}{4}\cot\Big(\frac{k_iL}{2}\Big), \quad \exp\Big(\sqrt{-1}\sum_{1\leqslant i\leqslant N} k_iL\Big)=1. \tag{3.2}$$

We investigate the real solutions for the first equation

$$\sin(q{+}\phi)-\lambda = \frac{U}{4}\cot\Big(\frac{qL}{2}\Big). \tag{3.3}$$

In the interval $0\leqslant q<2\pi$, its right hand side has L branches

$$\frac{2\pi}{L}\Big(\ell-\frac{1}{2}\Big) < q < \frac{2\pi}{L}\Big(\ell+\frac{1}{2}\Big), \quad \ell\in\Big\{\frac{2j-1}{2}\Big| j=1,2,\dots,L\Big\}. \tag{3.4}$$

Note that, with a given ℓ satisfying (3.4), the first equations in (3.1) are rewritten as

$$k_iL = 2\pi\ell - 2\arctan\Big(\frac{\lambda-\sin(k_i+\phi)}{U/4}\Big),$$

which are convenient to relate the ℓ to the (half-)integers appearing in the string hypothesis [28]. By regarding λ as a real parameter, we seek a solution q of (3.3) in one of the branches (3.4). From the graphical discussion (Figure 2), the solution in the branch ℓ is uniquely determined for arbitrary λ under the following condition:

$$-1 = \min_{0\leqslant q<2\pi}\frac{d}{dq}(\sin(q+\phi)-\lambda) > \max_{0\leqslant q<2\pi}\frac{d}{dq}\Big(\frac{U}{4}\cot\Big(\frac{qL}{2}\Big)\Big) = -\frac{UL}{8}.$$

Hence, for $U>\frac{8}{L}$, the solution of (3.3) can be written as an increasing function of λ, i.e., $q=q_\ell(\lambda)$. Given a non-repeating set $\{\ell_i|1\leqslant i\leqslant N\}\subset\{\frac{2j-1}{2}|1\leqslant j\leqslant L\}$ of the branches, the second equation in (3.2) is satisfied when $\frac{L}{2\pi}\sum_i q_{\ell_i}(\lambda)\in\mathbb{Z}$. The behaviour of the solution $q=q_\ell(\lambda)$ tells us that

$$\lim_{\lambda\to\pm\infty}\frac{L}{2\pi}\sum_{1\leqslant i\leqslant N} q_{\ell_i}(\lambda) = \sum_{1\leqslant i\leqslant N}\Big(\ell_i\pm\frac{1}{2}\Big). \tag{3.5}$$

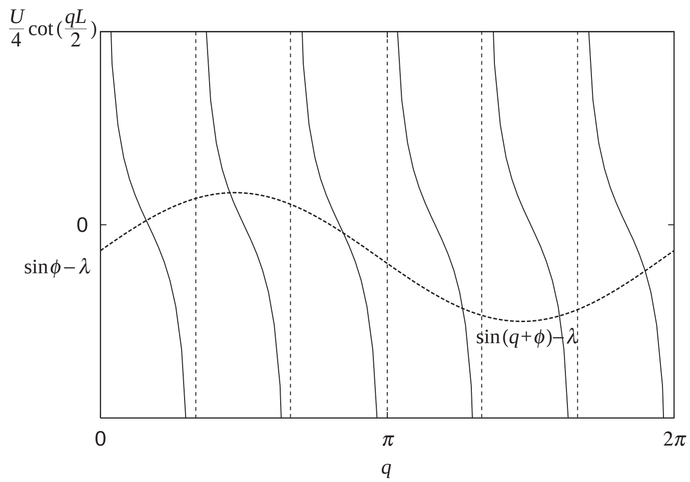

Figure 2: The generic behaviour of $\sin(q+\phi)-\lambda$ and $\frac{U}{4}\cot(\frac{qL}{2})$ in (3.3) under the condition $UL > 8$ for $L = 6$. Horizontal dashed lines divide six branches as specified by (3.4). Each branch has an intersection which leads to a solution of (3.3).

Thus there exist $N-1$ values of λ giving the following integer values for $\frac{L}{2\pi}\sum_i q_{\ell_i}(\lambda)$:

$$m \in \left\{ \sum_{1\leqslant i\leqslant N}\left(\ell_i - \frac{1}{2}\right) + j \,\middle|\, j = 1, 2, \ldots, N-1 \right\}.$$

Note that such $\{\lambda\}$ and integers $\{m\}$ are in one-to-one correspondence due to $\frac{dq_\ell(\lambda)}{d\lambda} > 0$. As a consequence, the solutions $\{k_i = q_{\ell_i}(\lambda), \lambda\}$ are specified by the set of indices $\{\ell_i, m\}$. The number of possible $\{\ell_i, m\}$ is given by $\binom{L}{N}(N-1)$. Here we note that the number is consistent with the formula $Z(L; N, M)$ in [3, 28].

Proposition 3.1. *For $U > \frac{8}{L}$, the Lieb-Wu equations* (2.3) *with $M = 1$ have $\binom{L}{N}(N-1)$ real solutions [2, 24].*

3.2 k-Λ-string solutions

Next we consider the solutions including a couple of complex wavenumbers. We assume the form of solutions as

$$k_i \in \mathbb{R}/2\pi\mathbb{R},\ (i = 1, 2, \ldots, N-2), \quad k_{N-1} = \zeta - \sqrt{-1}\xi, \quad k_N = \zeta + \sqrt{-1}\xi,$$

where $0 \leqslant \zeta < 2\pi$ and $\xi > 0$. Note that k_{N-1} and k_N form a complex conjugate pair which is referred to as k-Λ-2-string. Then the first set of equations in (3.1) are rewritten in terms

of real variables as follows

$$\sin(k_i+\phi) - \lambda = \frac{U}{4}\cot\left(\frac{k_i L}{2}\right), \quad (i = 1, 2, \ldots, N-2), \tag{3.6a}$$

$$\sin(\zeta+\phi)\cosh\xi - \lambda = \frac{U}{4}\frac{\sin(\zeta L)}{\cosh(\xi L) - \cos(\zeta L)}, \tag{3.6b}$$

$$\cos(\zeta+\phi)\sinh\xi = -\frac{U}{4}\frac{\sinh(\xi L)}{\cosh(\xi L) - \cos(\zeta L)}. \tag{3.6c}$$

On the other hand, the second equation in (3.1) is equivalent to the following condition:

$$\sum_{1\leqslant i\leqslant N-2} k_i + 2\zeta = \frac{2\pi}{L}m, \quad \text{with} \quad m = 0, 1, \ldots, NL-1. \tag{3.7}$$

In the same way as the previous case of section 3.2, if we consider a solution of each equation (3.6a) in one of the branches (3.4), and the solution can be written as a function of λ. Given a set $\{\ell_i | 1 \leqslant i \leqslant N-2\}$ of non-repeating indices specifying the branches (3.4), we express the solutions of (3.6a) as $k_i = q_{\ell_i}(\lambda), (1 \leqslant i \leqslant N-2)$. Then, from the relation (3.7), the ζ is also written as a function of λ,

$$\zeta = \zeta(\lambda) := \frac{\pi}{L}m - \frac{1}{2}\sum_{1\leqslant i\leqslant N-2} q_{\ell_i}(\lambda), \tag{3.8}$$

for fixed $\{\ell_i\}$ and m.

For an illustration, we consider (3.6b) and (3.6c) in the case $N = 2$. Since ζ does not depend on λ in the case, the equations (3.6b) and (3.6c) decouple into the following:

$$\lambda = \sin\left(\frac{\pi}{L}m+\phi\right)\cosh\xi, \quad \sinh\xi = -\frac{U}{4\cos(\frac{\pi}{L}m+\phi)}f^{(2)}(\xi), \tag{3.9}$$

where

$$f^{(2)}(\xi) := \frac{\sinh(\xi L)}{\cosh(\xi L) - (-1)^m} = \begin{cases} \coth(\xi L/2) & \text{for } m \in 2\mathbb{Z}, \\ \tanh(\xi L/2) & \text{for } m \in 2\mathbb{Z}+1. \end{cases}$$

We seek a solution of the second equation in (3.9) through graphical discussion. Since $f^{(2)}(\xi) > 0$ when $\xi > 0$, we need the condition $\frac{\pi}{2} < \frac{\pi}{L}m + \phi < \frac{3\pi}{2}$ so that the second equation of (3.9) has a solution. If such m is even, it is straightforward that the second equation in (3.9) determines a unique solution $\xi(>0)$ since $\lim_{\xi\to\infty} f^{(2)}(\xi) = 1$. For odd m, the equation has a unique solution if the condition

$$1 = \frac{d(\sinh\xi)}{d\xi}(0) < \min_{\substack{m\in 2\mathbb{Z}+1 \\ \frac{\pi}{2} < \frac{\pi}{L}m+\phi < \frac{3\pi}{2}}} \left(-\frac{U}{4\cos(\frac{\pi}{L}m+\phi)}\frac{df^{(2)}}{d\xi}(0)\right) = \frac{UL}{8},$$

is satisfied. The number of allowed values of m here depends on L and ϕ,

$$m \in \begin{cases} \left\{\frac{L}{2} - \frac{L}{\pi}\phi + j \middle| j = 1, 2, \ldots, L-1\right\}, & \frac{L}{2} - \frac{L}{\pi}\phi \in \mathbb{Z}, \\ \left\{\left\lfloor\frac{L}{2} - \frac{L}{\pi}\phi\right\rfloor + j \middle| j = 1, 2, \ldots, L\right\}, & \frac{L}{2} - \frac{L}{\pi}\phi \notin \mathbb{Z}. \end{cases}$$

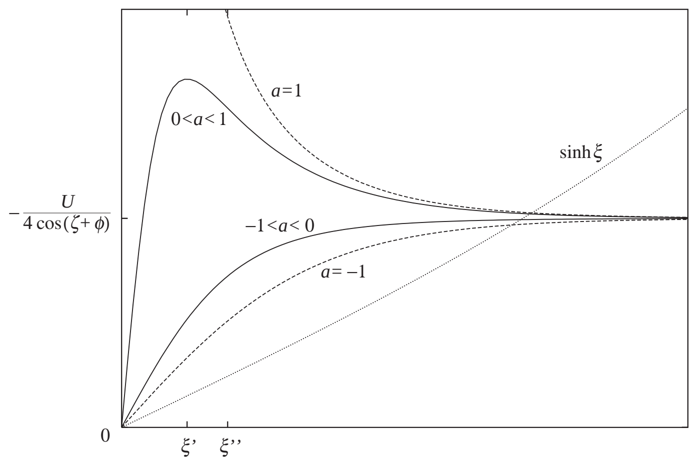

Figure 3: The behaviour of $\sinh\xi$ and $-\frac{U}{4\cos(\zeta+\phi)}f_a(\xi)$ in (3.10a) under the conditions $UL > 8$ and $\frac{\pi}{2} < \zeta+\phi < \frac{3\pi}{2}$. The ξ' and ξ'' respectively denote the maximum and the turning point of $-\frac{U}{4\cos(\zeta+\phi)}f_a(\xi)$ in the cases $0 < a < 1$

Let $\xi_m^{(2)}$ denote the solution specified by the m. By substituting the solution $\xi_m^{(2)}$ into the first equation in (3.9), one immediately obtains λ. Recall that, in the cases of $\frac{L}{2} - \frac{L}{\pi}\phi \in \mathbb{Z}$, i.e., even L,(respectively, odd L) and $\phi = 0$ or $\frac{\pi}{L}$, (respectively, $\phi = \frac{\pi}{2L}$ or $\frac{3\pi}{2L}$), the system has the $so(4)$ symmetry. Thus, as the $so(4)$ symmetry reduces to the spin-$sl(2)$ symmetry through the change of AB-flux strength ϕ, the number of solutions for the Lieb-Wu equations increases.

Let us consider the case $N > 2$. By inserting (3.8) into (3.6c), we have

$$\sinh\xi = -\frac{U}{4\cos(\zeta(\lambda)+\phi)}f_a(\xi), \tag{3.10a}$$

$$f_a(\xi) := \frac{\sinh(\xi L)}{\cosh(\xi L) - a}, \qquad a = (-)^m \cos\Big(\frac{L}{2}\sum_{1\leqslant i\leqslant N-2} q_{\ell_i}(\lambda)\Big). \tag{3.10b}$$

Note that $|a| \leqslant 1$. One sees that, for a fixed a, the function $f_a(\xi)$ has the following properties: i) $f_a(0) = 0$, $\frac{df_a}{d\xi}(0) \geqslant \frac{L}{2}$, $\lim_{\xi\to\infty} f_a(\xi) = 1$; ii) if $a < 0$, $f_a(\xi)$ is monotonically increasing and concave; iii) if $a > 0$, $f_a(\xi)$ has a single positive maximum at $\xi'(> 0)$ and a single turning point at $\xi''(> \xi')$. These properties are sufficient to discuss the solution ξ of (3.10) for arbitrary λ. From the graphical discussion similar to the case $N = 2$ (see Figure 3), this determines a unique ξ as a function of λ under the condition

$$1 = \frac{d(\sinh\xi)}{d\xi}(0) < \min_{\substack{|a|\leqslant 1 \\ \frac{\pi}{2}<\zeta+\phi<\frac{3\pi}{2}}} \Big(-\frac{U}{4\cos(\zeta+\phi)}\frac{df_a}{d\xi}(0)\Big) = \min_{|a|\leqslant 1}\frac{UL}{4(1-a)} = \frac{UL}{8},$$

if and only if $\frac{\pi}{2} < \zeta(\lambda)+\phi = \frac{\pi}{L}m+\phi-\frac{1}{2}\sum_i q_{\ell_i}(\lambda) < \frac{3\pi}{2}$. By using (3.5), it is sufficient to have

a unique solution for (3.10) that the integer m satisfies

$$\sum_{1\leqslant i\leqslant N-2}\Big(\ell_i+\frac{1}{2}\Big)+\frac{L}{2}-\frac{L}{\pi}\phi<m<\sum_{1\leqslant i\leqslant N-2}\Big(\ell_i-\frac{1}{2}\Big)+\frac{3L}{2}-\frac{L}{\pi}\phi,$$

that is,

$$m\in\begin{cases}\Big\{\sum_i\Big(\ell_i+\frac{1}{2}\Big)+\frac{L}{2}-\frac{L}{\pi}\phi+j\Big|j=1,2,\dots,L-N+1\Big\}, & \frac{L}{2}-\frac{L}{\pi}\phi\in\mathbb{Z},\\ \Big\{\sum_i\Big(\ell_i+\frac{1}{2}\Big)+\big\lfloor\frac{L}{2}-\frac{L}{\pi}\phi\big\rfloor+j\Big|j=1,2,\dots,L-N+2\Big\}, & \frac{L}{2}-\frac{L}{\pi}\phi\notin\mathbb{Z}.\end{cases}\tag{3.11}$$

Note that $\lim_{\lambda\to\pm\infty}\xi(\lambda)=\xi^{(2)}_{m-\sum_i(\ell_i\pm\frac{1}{2})}$, which is well-defined for the above m. We see that, for the values of m given in (3.11), the equation (3.6b) with $\xi(\lambda)$ and $\zeta(\lambda)$

$$\begin{aligned}\lambda&=\sin\big(\tfrac{\pi}{L}m+\phi-\tfrac{1}{2}\textstyle\sum_i q_{\ell_i}(\lambda)\big)\cosh\big(\xi(\lambda)\big)-\frac{U}{4}\frac{\sin(\frac{L}{2}\sum_i q_{\ell_i}(\lambda))}{\cos\big(\frac{L}{2}\sum_i q_{\ell_i}(\lambda)\big)-(-)^m\cosh\big(\xi(\lambda)L\big)}\\&=:g\big(\{q_{\ell_i}(\lambda)\},\xi(\lambda)\big),\end{aligned}\tag{3.12}$$

determines λ. In fact, since $q_{\ell_i}(\lambda)$ and $\xi(\lambda)$ are continuous functions of λ and the function g satisfies the following:

$$\lim_{\lambda\to\pm\infty}g\big(\{q_{\ell_i}(\lambda)\},\xi(\lambda)\big)=g\Big(\{\tfrac{2\pi}{L}(\ell_i\pm\tfrac{1}{2})\},\xi^{(2)}_{m-\sum_i(\ell_i\pm\frac{1}{2})}\Big).$$

Here g is a continuous and finite function with respect to λ. Hence there exists a solution λ for the equation (3.12).

Proposition 3.2. *For $U>\frac{8}{L}$, the Lieb-Wu equations* (2.3) *with $M=1$ have $\binom{L}{N-2}(L-N+1)$ k-Λ-2-string solutions if the system has the $so(4)$ symmetry, and they have $\binom{L}{N-2}(L-N+2)$ k-Λ-2-solutions, otherwise.*

One notices that, only for the system with the $so(4)$ symmetry, the number of k-Λ-2-solutions is consistent with the string hypothesis [28]. Note that, for $0<U<\frac{8}{L}$, some of the k-Λ-2-strings may disappear. In Appendix A, we numerically investigate the case $N=2$ and show that, for $0<U<\frac{8}{L}$, the k-Λ-2-strings with odd m disappear, while additional real solutions appear [5]. For the system with only the spin-$sl(2)$ symmetry, the Lieb-Wu equations have more k-Λ-2-solutions than those expected by the string hypothesis.

3.3 Completeness of Bethe ansatz for $L=3$

Applying the above results, we now show that all the eigenstates can be constructed through the Bethe ansatz method in a simple case. We consider the case $L=3$ and $\phi\neq\frac{\pi}{2L},\frac{3\pi}{2L}$ when the system does not have $so(4)$ symmetry but the spin-$sl(2)$ symmetry. We note that the completeness of eigenstates in this situation has not been discussed in the literature.

The number of Bethe states $|k,\lambda;s\rangle^{\phi}_{N,M}, (0 \leqslant 2M \leqslant N \leqslant 3)$ is exactly calculated as follows: the case $M = 0$ is trivial since the eigenstates are those of lattice free fermion system; for the cases $(N,M) = (2,1)$ and $(3,1)$, we have obtained the following formulas from Proposition 3.1 and 3.2:

$$
\sharp(\text{Bethe states}) = \begin{cases} \begin{pmatrix} 3 \\ N \end{pmatrix} \begin{pmatrix} N-1 \\ 1 \end{pmatrix} & \text{for real solutions,} \\ \begin{pmatrix} 3 \\ N-2 \end{pmatrix} \begin{pmatrix} 5-N \\ 1 \end{pmatrix} & \text{for } k\text{-}\Lambda\text{-2-string solutions.} \end{cases}
$$

Since each Bethe state $|k,\lambda;s\rangle^{\phi}_{N,M}$ corresponds to the highest weight vector of a highest weight $sl(2)$-module, we should count their $sl(2)$-descendant states

$$
(S_-)^n |k,\lambda;s\rangle^{\phi}_{N,M}, \qquad (0 < n \leqslant N - 2M).
$$

The eigenstates with $4 \leqslant N \leqslant 6$, which are not Bethe states nor their $sl(2)$-descendant states, are constructed through the transformation $T_{\rm r}T_{\rm ph}$ as we have described in Section 2.3. Indeed, by applying $T_{\rm r}T_{\rm ph}$ to the Bethe state $|k,\lambda;s\rangle^{\phi+\pi}_{6-N,3-M}, (4 \leqslant N \leqslant 2M \leqslant 6)$ of the system described by the Hamiltonian $H_{\phi+\pi}$, we get the eigenstate $T_{\rm r}T_{\rm ph}|k,\lambda;s\rangle^{\phi+\pi}_{6-N,3-M}$ of H_ϕ with $4 \leqslant N \leqslant 2M \leqslant 6$ which is the lowest weight vector of a highest weight $sl(2)$-module. We also count the eigenstates

$$
(S_+)^n T_{\rm r}T_{\rm ph}|k,\lambda;s\rangle^{\phi+\pi}_{6-N,3-M}, \qquad (0 < n \leqslant 2M - N).
$$

Table 3.3 indeed shows that we obtain $64 = 4^3 = \dim V$ eigenstates, which give a complete system of the Fock space V.

4 Combinatorial completeness of Bethe ansatz

The Bethe ansatz method was first introduced in the case of one-dimensional spin-$\frac{1}{2}$ isotropic Heisenberg spin chain [1]. Bethe assumed the string hypothesis and estimated the number $Z(N;M)$ of solutions for the Bethe equations with M down-spins on an N-site chain as

$$
Z(N;M) = \sum_{\substack{\{M_n\} \\ M=\sum nM_n}} \prod_{m \geqslant 1} \begin{pmatrix} P_m + M_m \\ M_m \end{pmatrix}, \tag{4.1}
$$

where M_n denotes the number of n-strings composing a solution and $P_n = N - 2M + 2\sum_{m(>n)}(m-n)M_m$. He obtained the following summation formula:

$$
Z(N;M) = \begin{pmatrix} N \\ M \end{pmatrix} - \begin{pmatrix} N \\ M-1 \end{pmatrix}. \tag{4.2}
$$

N	M	$6-N$	$3-M$	type of solutions	$\sharp$(Bethe)	$sl(2)$ sym.	$\sharp$(state)
0	0			real	1	1	1
1	0			real	3	2	6
2	0			real	3	3	9
2	1			real	3	1	3
2	1			k-Λ-2-string	3	1	3
3	0			real	1	4	4
3	1			real	2	2	4
3	1			k-Λ-2-string	6	2	12
4	2	2	1	real	3	1	3
4	2	2	1	k-Λ-2-string	3	1	3
4	3	2	0	real	3	3	9
5	3	1	0	real	3	2	6
6	3	0	0	real	1	1	1
							64

Table 1: Enumeration of eigenstates for $L = 3$ and $\phi \neq \frac{\pi}{2L}, \frac{3\pi}{2L}$.

which implies that the number $Z(N; M)$ of Bethe states is interpreted as the multiplicity of (N–$2M$+1)-dimensional irreducible $sl(2)$-modules in the tensor product of N two-dimensional irreducible $sl(2)$-modules. By taking into account that the Bethe states are $sl(2)$-highest and generate $(N-2M+1)$ $sl(2)$-descendant states, the completeness of Bethe ansatz is shown in a combinatorial way as

$$\sum_{0 \leqslant M \leqslant \lfloor N/2 \rfloor} (N-2M+1) Z(N; M) = 2^N.$$

where $\lfloor x \rfloor$ denotes the greatest integer in x.

It is known that, in general, solutions of Bethe equations do not have the nature assumed in the string hypothesis [4, 8, 9]. Indeed, for $N > 21$, some of the 2-string solutions are redistributed to real solutions that are not counted in $Z(N; M)$ [4]. Hence, when we actually employ the the formula $Z(N; M)$ to show the completeness of Bethe ansatz, we must regard such redistributed real solutions as 2-string solutions.

We apply the techniques developed in [12, 13] to the Hubbard model with $so(4)$ symmetry. Indeed a new proof for Essler-Korepin-Schoutens' combinatorial completeness of Bethe ansatz [3] is obtained as a corollary of the relation among the characters of $so(4)$-modules. Moreover, based on the results in the previous section, we propose the conjectural formula related to the combinatorial completeness of Bethe ansatz for the system with only the charge-

$u(1)$ and spin-$sl(2)$ symmetry. In both cases, we obtain the formulas corresponding to (4.2), which has not been established even for the $so(4)$-case.

4.1 Kirillov's power series and Q-system

First we give three lemmas introduced in the case of one-dimensional isotropic Heisenberg spin chain [12, 13]. Detailed proofs are given in [12, 13]. Let $a_1, a_2, \ldots, a_l$ be a set of integers for $l \gg L$. Define a set of formal power series $\{\varphi_n(z_n, z_{n+1}, \ldots, z_l) | 1 \leqslant n \leqslant l\}$ by

$$\begin{aligned}\psi_n(z) &:= (1-z)^{-a_n+2M-1}, \\ \varphi_n(z_n, z_{n+1}, \ldots, z_l) &:= \psi_n(z_n)\varphi_{n+1}((1-z_n)^{-2}z_{n+1}, \ldots, (1-z_n)^{-2(l-n)}z_l).\end{aligned}$$

Lemma 4.1. *The power series $\varphi_n(z_n, \ldots, z_l)$ has the following expression:*

$$\varphi_n(z_n, \ldots, z_l) = \sum_{M_n, \ldots, M_l \geqslant 0} \prod_{n \leqslant m \leqslant l} \binom{\mathcal{P}_m(a_m)+M_m}{M_m} z_n^{M_n} \cdots z_l^{M_l}, \qquad \textit{for} \quad 1 \leqslant n \leqslant l,$$

where

$$\mathcal{P}_n(a_n) = a_n - 2M + 2\sum_{m>n}(m-n)M_m.$$

Proof. The case $n = l$ is given by the formula,

$$(1-z)^{-\alpha-1} = \sum_{m \geqslant 0} \binom{\alpha+m}{m} z^m.$$

Then the case $1 \leqslant n < l$ is proved by induction on n. □

Introduce the variables $\{z_n^{(k)} | 1 \leqslant n \leqslant l, 0 \leqslant k \leqslant n\}$ through

$$z_n^{(0)} = z_n, \qquad z_n^{(k)} = (1 - z_k^{(k-1)})^{-2(n-k)} z_n^{(k-1)}, \quad \text{for} \quad 1 \leqslant k \leqslant n.$$

Lemma 4.2. *The power series $\varphi_1(z_1, \ldots, z_l)$ is rewritten in terms of the variables $\{z_n^{(k)}\}$,*

$$\varphi_1(z_1, \ldots, z_l) = \prod_{1 \leqslant n \leqslant l} (1 - z_n^{(n-1)})^{-a_n+2M-1}.$$

Proof. Since

$$\begin{aligned}\varphi_k(z_k^{(k-1)}, \ldots, z_l^{(k-1)}) &= \psi_k(z_k^{(k-1)})\varphi_{k+1}((1-z_k^{(k-1)})^{-2}z_{k+1}^{(k-1)}, \ldots, (1-z_k^{(k-1)})^{-2(l-k)}z_l^{(k-1)}) \\ &= (1 - z_k^{(k-1)})^{-a_k+2M-1}\varphi_{k+1}(z_{k+1}^{(k)}, \ldots, z_l^{(k)}),\end{aligned}$$

for $1 \leqslant k \leqslant l-1$, the lemma is proved. □

Define the polynomials $\{Q_n = Q_n(t) | n \in \mathbb{Z}_{\geqslant 0}\}$ through the recursion relation,

$$Q_{n+2} = Q_{n+1} - tQ_n, \quad Q_0 = Q_1 = 1. \tag{4.3}$$

Lemma 4.3.

$$i)\quad Q_n^2 = Q_{n+1}Q_{n-1} + t^n, \quad \textit{for} \quad n \geqslant 1, \tag{4.4a}$$

$$ii)\quad \varphi_1(t, t^2, \dots, t^l) = \prod_{1\leqslant n\leqslant l} (Q_{n+1}Q_n^{-2}Q_{n-1})^{-a_n+2M-1}, \tag{4.4b}$$

$$iii)\quad Q_n(t(v)) = \frac{(1-v)^{n-1}(1-v^{n+1})}{(1-v^2)^n}, \quad \textit{where} \quad t(v) := \frac{v}{(1+v)^2}. \tag{4.4c}$$

Proof. i) Use induction on n.
ii) Set $z_n^{(0)} = z_n = t^n, (1 \leqslant n \leqslant l)$ in $\{z_n^{(k)}\}$. One obtains the following relations:

$$1 - z_k^{(k-1)} = Q_{k+1}Q_k^2 Q_{k-1}, \quad z_n^{(k)} = Q_{k+1}^{-2(n-k)} Q_k^{2(n-k-1)} t^n, \quad \text{for} \quad 1 \leqslant k \leqslant n.$$

Combining these with Lemma 4.2, we can prove (4.4b).
iii) One can directly verify that the $Q_n(t(v))$ satisfies the recursion relation (4.3) with $t = t(v)$. □

The relations (4.4a) are called the Q-system of type $sl(2)$, which is a key object in [12]. Indeed the expression (4.4b) produces an identity among the characters of $sl(2)$-modules. The Q-system also plays a significant role in the combinatorial identities associated with the XXZ-Heisenberg spin chain and its generalizations [14, 17–19].

4.2 Combinatorial formulas

Using the above lemmas, we discuss the Hubbard-case. In the similar way, we define $\varphi'_n(z_n, z_{n+1}, \dots, z_l)$ and $\mathcal{P}'_n(a'_n)$ by replacing a_n and $2M$ with a'_n and N in $\varphi_n(z_n, z_{n+1}, \dots, z_l)$ and $\mathcal{P}_n(a_n)$, respectively. Define

$$\varphi(s,t) := (1+s)^L \varphi'_1(s^2 t, s^4 t^2, \dots, s^{2l} t^l) \varphi_1(t, t^2, \dots, t^l).$$

The following is straightforward from Lemma 4.1:

$$\varphi(s,t) = \sum_{\substack{0\leqslant N_r\leqslant L\\ \{M_n, M'_n\}}} \binom{L}{N_r} \prod_{1\leqslant n\leqslant l} \binom{\mathcal{P}'_n(a'_n)+M'_n}{M'_n} \binom{\mathcal{P}_n(a_n)+M_n}{M_n} s^{N_r+2\sum_{m\geqslant 1} mM'_m} t^{\sum_{m\geqslant 1} m(M_m+M'_m)}.$$

Then the coefficient of $s^N t^M, (0 \leqslant 2M \leqslant N \leqslant L)$ in $\varphi(s,t)$ is expressed by

$$\mathcal{Z}(L, \{a_n, a'_n\}; N, M) := \sum_{\substack{\{N_r, M_n, M'_n\}\\ N=N_r+2\sum nM'_n\\ M=\sum n(M_n+M'_n)}} \binom{L}{N_r} \prod_{1\leqslant n\leqslant l} \binom{\mathcal{P}'_n(a'_n)+M'_n}{M'_n} \binom{\mathcal{P}_n(a_n)+M_n}{M_n}.$$

where the sum runs over all configurations $\{N_r, M_n, M'_n \geqslant 0\}$ such that $N = N_r + 2\sum_{n\geqslant 0} nM'_n$ and $M = \sum_{n\geqslant 1} n(M_n + M'_n)$. We calculate explicit forms for the power series $\varphi(s,t)$ after taking the special values of $\{a_n\}$ and $\{a'_n\}$.

Introduce

$$Z(L;N,M)=\sum_{\substack{\{N_r,M_n,M'_n\}\\ N=N_r+2\sum nM'_n\\ M=\sum n(M_n+M'_n)}}\binom{L}{N_r}\prod_{n\geqslant 1}\binom{P'_n+M'_n}{M'_n}\binom{P_n+M_n}{M_n},\tag{4.5a}$$

$$\widetilde{Z}(L;N,M)=\sum_{\substack{\{N_r,M_n,M'_n\}\\ N=N_r+2\sum nM'_n\\ M=\sum n(M_n+M'_n)}}\binom{L}{N_r}\prod_{n\geqslant 1}\binom{P'_n+M'_n+n}{M'_n}\binom{P_n+M_n}{M_n},\tag{4.5b}$$

where

$$P'_n=L-N+2\sum_{m>n}(m-n)M'_m,\qquad P_n=N-2M+2\sum_{m>n}(m-n)M_m.$$

Note that $P'_n, P_n\geqslant 0$ due to $0\leqslant 2M\leqslant N\leqslant L$.

The $Z(L;N,M)$ (4.5a) is the very number of Bethe states for the Hubbard model estimated under the string hypothesis [3, 28, 30]. In terms of the string hypothesis, N_r denotes the number of real k's, M_n the number of Λ-n-strings, and M'_n the number of k-Λ-$2n$-strings. We have verified in the previous section that, if the system has $so(4)$ symmetry, the number

$$Z(L;N,1)=\binom{L}{N}\binom{N-1}{1}+\binom{L}{N-2}\binom{L-N+1}{1},$$

gives the correct number of solutions for the Lieb-Wu equations (2.3) with $M=1$. In fact the first and second terms in the above $Z(L;N,1)$ correspond to the following two cases: N real k's ($N_r=N$) with a Λ-1-string ($M_1=1$), and $N-2$ real k's ($N_r=N-2$) with a k-Λ-2-string ($M'_1=1$), respectively.

We now propose the $\widetilde{Z}(L;N,M)$ (4.5b) as a formula counting the number of Bethe states for the system with charge-$u(1)$ and spin-$sl(2)$ symmetries. Indeed

$$\tilde{Z}(L;N,1)=\binom{L}{N}\binom{N-1}{1}+\binom{L}{N-2}\binom{L-N+2}{1},$$

is consistent with the number and the string-type of solutions for the Lieb-Wu equations (2.3) with $M=1$ in the $sl(2)$-case. The first term corresponds to the case of N real k's ($N_r=N$) with a Λ-1-string ($M_1=1$), and the second term to the case of $N-2$ real k's ($N_r=N-2$) with a k-Λ-2-string ($M'_1=1$). We note that, for $L<N\leqslant 2L$, we interpret $\widetilde{Z}(L;N,M)$ as the number of the lowest weight vectors $T_{\mathrm{r}}T_{\mathrm{ph}}|k,\lambda;s\rangle^{\phi+\pi}_{2L-N,L-M}$. To derive $\tilde{Z}(L;N,M)$ (4.5b) for $M\geqslant 2$ from Takahashi's string center equations [28, 30], we need to appropriately extend the region of the allowed (half-)integers characterizing the Bethe states.

In both the $so(4)$- and $sl(2)$-cases, one must also take a redistribution phenomenon into consideration [3, 4, 8, 9]; what it means here will be more clear in Appendix A.

Proposition 4.4. *We have the following identities:*

$$
\begin{aligned}
&i) \quad (1+u)^L(1+uv)^L(1-u^2v)(1-v) = \sum_{\substack{0\leqslant N\leqslant 2L+2\\ 0\leqslant M\leqslant L+1}} Z(L;N,M)u^Nv^M,\\
&ii) \quad (1+u)^L(1+uv)^L(1-v) = \sum_{\substack{0\leqslant N\leqslant 2L\\ 0\leqslant M\leqslant L+1}} \widetilde{Z}(L;N,M)u^Nv^M.
\end{aligned}
\tag{4.6}
$$

Proof. By using Lemma 4.3, we have

$$
\varphi(s,t) = (1+s)^L \prod_{1\leqslant n\leqslant l} (Q'_{n+1}Q'^{-2}_nQ'_{n-1})^{-a'_n+N-1}(Q_{n+1}Q_n^{-2}Q_{n-1})^{-a_n+2M-1},
$$

where $Q'_n(t) := Q_n(s^2t)$. Through the change of variables

$$
s = \frac{u(1+v)}{1+u^2v}, \quad t = \frac{v}{(1+v)^2}, \quad ds\,dt = \frac{(1-u^2v)(1-v)}{(1+u^2v)^2(1+v)^2}du\,dv,
$$

the coefficient of s^Nt^M in $\varphi(s,t)$ is calculated as

$$
\begin{aligned}
&\mathcal{Z}(L,\{a_n,a'_n\};N,M) = \operatorname*{Res}_{s=0,t=0} \varphi(s,t)\frac{ds}{s^{N+1}}\frac{dt}{t^{M+1}}\\
&= \operatorname*{Res}_{u=0,v=0}\Big(\frac{(1+u)^L(1+uv)^L}{(1+u^2v)^L}\prod_{1\leqslant n\leqslant l}(Q'_{n+1}Q'^{-2}_nQ'_{n-1})^{-a'_n+N-1}(Q_{n+1}Q_n^{-2}Q_{n-1})^{-a_n+2M-1}\\
&\quad\times\frac{(1+u^2v)^{N+1}}{(1+v)^{N+1}}(1+v)^{2M+2}\frac{(1-u^2v)(1-v)}{(1+u^2v)^2(1+v)^2}\Big)\frac{du}{u^{N+1}}\frac{dv}{v^{M+1}}\\
&= \operatorname*{Res}_{u=0,v=0}\Big(\frac{(1+u)^L(1+uv)^L}{(1+u^2v)^L}\frac{(1-u^2v)(1-v)}{(1+v)^N}\prod_{1\leqslant n\leqslant l}(1-u^{2n}v^n)^{b'_n}(1-v^n)^{b_n}\Big)\frac{du}{u^{N+1}}\frac{dv}{v^{M+1}},
\end{aligned}
$$

where we have introduced

$$
b_n = -a_n + 2a_{n-1} - a_{n-2}, \qquad b'_n = -a'_n + 2a'_{n-1} - a'_{n-2}, \qquad (1\leqslant n\leqslant l),
$$

with $a_{-1} = a_0 = a'_{-1} = a'_0 = 0$. Note that, in the third equality, we have employed the assumption $l \gg L$. Setting $a_{n\geqslant 1} = L$ and $a'_{n\geqslant 1} = N$, we have $-b'_1 = b'_2 = L, -b_1 = b_2 = N$ and $b'_{n\geqslant 3} = b'_{n\geqslant 3} = 0$, which proves the identity i) in (4.6). And, setting $a_{n\geqslant 1} = L+n$ and $a'_{n\geqslant 1} = N$, we have $b'_1 = -L-1, b'_2 = L, -b_1 = b_2 = N$ and $b'_{n\geqslant 3} = b'_{n\geqslant 3} = 0$, which proves the identity ii). □

Through the change of variables $u = xy^{-1}$ and $v = y^2$, we have

$$
(x+x^{-1}+y+y^{-1})^L(x-x^{-1})(y-y^{-1}) = \sum_{\substack{0\leqslant N\leqslant 2L+2\\ 0\leqslant M\leqslant L+1}} Z(L;N,M)x^{-L+N-1}y^{-N+2M-1}, \tag{4.7a}
$$

$$
(x+x^{-1}+y+y^{-1})^L(y-y^{-1}) = \sum_{\substack{0\leqslant N\leqslant 2L\\ 0\leqslant M\leqslant L+1}} \widetilde{Z}(L;N,M)x^{-L+N}y^{-N+2M-1}. \tag{4.7b}
$$

First we consider the relation (4.7a). If we express the left-hand side of (4.7a) as $F(x,y)$, we find the property $F(x^{-1},y) = F(x,y^{-1}) = -F(x,y)$. Then the first relation (4.7a) is rewritten as

$$(x+x^{-1}+y+y^{-1})^L = \sum_{\substack{0\leqslant N\leqslant L \\ 0\leqslant M\leqslant \lfloor N/2\rfloor}} Z(L;N,M)\frac{x^{L-N+1}-x^{-L+N-1}}{x-x^{-1}}\frac{y^{N-2M+1}-y^{-N+2M-1}}{y-y^{-1}}, \quad (4.8)$$

where $\lfloor x\rfloor$ denotes the greatest integer in x. Let V'_n be an $(n{+}1)$-dimensional irreducible $sl(2)$-module associated with the charge-$sl(2)$ symmetry and V_n that associated with the spin-$sl(2)$ symmetry. We introduce an $(n{+}1)(m{+}1)$-dimensional irreducible $so(4)$-module by the tensor product $V_{n,m} = V'_n \otimes V_m$. Through the representation (2.5) and (2.6) of $so(4)$, the Fock space V of the L-site system is isomorphic to the tensor product $(V_{1,0}\oplus V_{0,1})^{\otimes L}$ as an $so(4)$-module. Note that

$$\boldsymbol{\eta}^2|v\rangle = \frac{n}{2}\left(\frac{n}{2}+1\right)|v\rangle, \quad \boldsymbol{S}^2|v\rangle = \frac{m}{2}\left(\frac{m}{2}+1\right)|v\rangle, \qquad \text{for } |v\rangle \in V_{n,m}\subset V.$$

We now decompose the Fock space V into the direct sum of $V_{n,m}$. From the relations (2.8), each Bethe state $|k,\lambda;s\rangle_{N,M}$ corresponds to the highest weight vector belonging to $V_{L-N,N-2M} = V_{2\eta,2S}$. The characters of the $so(4)$-module $V_{n,m}$ are calculated as

$$\mathrm{ch}\, V_{n,m} = \frac{x^{n+1}-x^{-n-1}}{x-x^{-1}}\frac{y^{m+1}-y^{-m-1}}{y-y^{-1}}.$$

To be precise, $x = \mathrm{e}^{\Lambda'_1}$ and $y = \mathrm{e}^{\Lambda_1}$ where both Λ'_1 and Λ_1 are the fundamental weight of $sl(2)$ and they are orthogonal to each other. One notices that, in terms of the characters, the identity (4.8) can be rewritten as

$$(\mathrm{ch}\, V_{1,0} + \mathrm{ch}\, V_{0,1})^L = \sum_{\substack{0\leqslant N\leqslant L \\ 0\leqslant M\leqslant \lfloor N/2\rfloor}} Z(L;N,M)\,\mathrm{ch}\, V_{L-N,N-2M}. \quad (4.9)$$

Theorem 4.5 (Multiplicity formula). *The multiplicity of the irreducible component $V_{L-N,N-2M}$ in the tensor product $(V_{1,0}\oplus V_{0,1})^{\otimes L}$ is given by $Z(L;N,M)$,*

$$(V_{1,0}\oplus V_{0,1})^{\otimes L} = \bigoplus_{\substack{0\leqslant N\leqslant L \\ 0\leqslant M\leqslant \lfloor N/2\rfloor}} Z(L;N,M) V_{L-N,N-2M}. \quad (4.10)$$

Next we turn to the relation (4.7b). If we express the left-hand side of (4.7b) as $\widetilde{F}(x,y)$, we find $\widetilde{F}(x,y^{-1}) = -\widetilde{F}(x,y)$. This gives

$$\left(1+x(y+y^{-1})+x^2\right)^L = \sum_{\substack{0\leqslant N\leqslant 2L \\ 0\leqslant M\leqslant \lfloor N/2\rfloor}} \widetilde{Z}(L;N,M)\, x^N\, \frac{y^{N-2M+1}-y^{-N+2M-1}}{y-y^{-1}}. \quad (4.11)$$

Let $V^{(N)} \subset V$ be a subspace of the Fock space V with N electrons and let $V_m^{(N)} \subset V^{(N)}$ be an $(m+1)$-dimensional irreducible $sl(2)$-module related to the spin-$sl(2)$ symmetry in $V^{(N)}$. The $V_m^{(N)}$ can be regarded as a $1\times(m+1)$-dimensional irreducible $\big(u(1)\oplus sl(2)\big)$-module by employing the representation (2.5) of $sl(2)$ and taking $u(1) = \mathbb{C}(L-2\eta_z)$ with (2.6). Here

$$(L-2\eta_z)|v\rangle = N|v\rangle, \quad \boldsymbol{S}^2|v\rangle = \frac{m}{2}\left(\frac{m}{2}+1\right)|v\rangle, \qquad \text{for } |v\rangle \in V_m^{(N)} \subset V.$$

The Fock space V of the L-site Hubbard model is isomorphic to the tensor product $(V_0^{(0)} \oplus V_1^{(1)} \oplus V_0^{(2)})^{\otimes L}$ as a $\big(u(1)\oplus sl(2)\big)$-module. We decompose the Fock space V into the direct sum of $V_m^{(N)}$. From the first relation in (2.8), each Bethe state $|k,\lambda;s\rangle_{N,M}$ is the highest weight vector belonging to $V_{N-2M}^{(N)} = V_{2S}^{(N)}$. The characters of $V_m^{(N)}$ are calculated as

$$\operatorname{ch} V_m^{(N)} = x^N\,\frac{y^{m+1}-y^{-m-1}}{y-y^{-1}}.$$

In terms of the characters, the identity (4.11) can be rewritten as

$$(\operatorname{ch} V_0^{(0)} + \operatorname{ch} V_1^{(1)} + \operatorname{ch} V_0^{(2)})^L = \sum_{\substack{0\leqslant N\leqslant 2L\\ 0\leqslant M\leqslant \lfloor N/2\rfloor}} Z(L;N,M)\operatorname{ch} V_{N-2M}^{(N)}. \tag{4.12}$$

Theorem 4.6 (Multiplicity formula). *The multiplicity of the irreducible component $V_{N-2M}^{(N)}$ in the tensor product $(V_0^{(0)} \oplus V_1^{(1)} \oplus V_0^{(2)})^{\otimes L}$ is given by $\widetilde{Z}(L;N,M)$,*

$$(V_0^{(0)} \oplus V_1^{(1)} \oplus V_0^{(2)})^{\otimes L} = \bigoplus_{\substack{0\leqslant N\leqslant 2L\\ 0\leqslant M\leqslant \lfloor N/2\rfloor}} \widetilde{Z}(L;N,M) V_{N-2M}^{(N)}. \tag{4.13}$$

Corollary 4.7 (Combinatorial completeness). *We have*

$$\text{i)} \quad \dim V = \sum_{\substack{0\leqslant N\leqslant L\\ 0\leqslant M\leqslant \lfloor N/2\rfloor}} (L-N+1)(N-2M+1)Z(L;N,M),$$

$$\text{ii)} \quad \dim V = \sum_{\substack{0\leqslant N\leqslant 2L\\ 0\leqslant M\leqslant \lfloor N/2\rfloor}} (N-2M+1)\widetilde{Z}(L;N,M). \tag{4.14}$$

Proof. Consider the limit $x,y\to 1$ in the identities (4.8) and (4.11). □

The identity i) in Corollary 4.7 reproduces the combinatorial completeness of Bethe states for the Hubbard model with $so(4)$ symmetry obtained by Essler, Korepin and Schoutens [3]. The factor $(L-N+1)(N-2M+1)$ in i) corresponds to the dimension of the highest weight $so(4)$-module $V_{L-N,N-2M}$ with the highest weight vector $|k,\lambda;s\rangle_{N,M}$.

In Essler-Korepin-Schoutens' proof of i) in Corollary 4.7, they take the sum on N after taking that on M. In our proof, the sums on N and M are taken "simultaneously" in the level of characters.

The factor $(N-2M+1)$ of the identity ii) in Corollary 4.7 for $0 \leqslant N \leqslant L$ is the dimension of the highest weight $\big(u(1) \oplus sl(2)\big)$-module $V_{N-2M}^{(N)}$ with the highest weight vector $|k, \lambda; s\rangle_{N,M}$. For $L < N \leqslant 2L$, the factor $(N-2M+1)$ should be interpreted as the dimension of the highest weight $\big(u(1) \oplus sl(2)\big)$-module $V_{N-2M}^{(N)}$ with the lowest weight vector $T_{\mathrm{r}}T_{\mathrm{ph}}|k, \lambda; s\rangle_{2L-N,L-M}^{\phi+\pi}$. If even L, the identity ii) can be rewritten as

$$\dim V = \sum_{\substack{0 \leqslant N \leqslant L \\ 0 \leqslant M \leqslant \lfloor N/2 \rfloor}} 2^{1-\delta_{N,L}} (N-2M+1) \widetilde{Z}(L; N, M),$$

by considering the particle-hole symmetry of the system (2.7). Thus we speculate that the identity ii) in Corollary 4.7 accounts for the combinatorial completeness of Bethe states for the system with the charge-$u(1)$ and the spin-$sl(2)$ symmetries.

The identities (4.7) also enabled us to get the explicit form of $Z(L; N, M)$ through the binomial theorem.

Corollary 4.8. *We obtain the summation formulas for $Z(L; N, M)$ and $\widetilde{Z}(L; N, M)$,*

$$\begin{aligned} &i) \quad Z(L; N, M) = \binom{L+2}{M}\binom{L}{N-M} - \binom{L}{M-1}\binom{L+2}{N-M+1}, \\ &ii) \quad \widetilde{Z}(L; N, M) = \binom{L}{M}\binom{L}{N-M} - \binom{L}{M-1}\binom{L}{N-M+1}. \end{aligned} \tag{4.15}$$

5 Summary and concluding remarks

In the framework of Bethe ansatz, we have studied the Hubbard model with the AB-flux that controls the symmetry of the system. In Section 3 we have shown the existence of solutions for Lieb-Wu equations with an arbitrary number of up-spins and one down-spin. We have found that the number of k-Λ-2-solutions increases as the $so(4)$ symmetry reduces to the spin-$sl(2)$ symmetry (Proposition 3.2). The number of Bethe states is consistent with the string hypothesis only in the $so(4)$-case. In Section 4 we have investigated the combinatorial formulas giving the combinatorial completeness of Bethe states. We have shown that the number of Bethe states can be interpreted as the multiplicity of irreducible components in the tensor products of $so(4)$-modules (Theorem 4.5). Essler-Korepin-Schoutens' combinatorial formula is reproduced by the relation (4.9) among the characters of $so(4)$-modules (Corollary 4.7). An advantage of our approach is that we can obtain the summation formula (4.15) for $Z(L; N, M)$. We have also proposed a new combinatorial formula derived from the relation (4.12) among the characters of $\big(u(1) \oplus sl(2)\big)$-modules. The formula is related to the combinatorial completeness of Bethe ansatz in the $sl(2)$-case (Corollary 4.7). It should be remarked that, in Section 3, we have not proved the uniqueness of solutions. Although the Lieb-Wu equations may have

other solutions that we have not expected, the combinatorial formulas introduced in Section 4 give an evidence that, for $M = 1$, we have found solutions enough to verify the combinatorial completeness of Bethe ansatz. The problem is open for $M \geqslant 2$.

The combinatorial completeness of Bethe ansatz has not been discussed for the one-dimensional isotropic Heisenberg spin chain with twisted boundary conditions. Kirillov's identity [12] also produces the following formula:

$$\tilde{Z}(N; M) = \sum_{\substack{\{M_n\} \\ M=\sum nM_n}} \prod_{n\geqslant 1} \binom{P_n+M_n+n}{M_n} = \binom{N}{M},$$

where $P_n = N - 2M + 2\sum_{m(>n)}(m - n)M_m$. It is expected that, if the redistribution phenomenon of solutions for Bethe equations [4, 8, 9] is taken into consideration, the formula corresponds to the system with twisted boundary conditions. We remark that the formula also appears in the different context [20].

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A Redistribution phenomenon

In Section 3, we exactly show the existence of solutions of Lieb-Wu equations with $M = 1$ for $U > \frac{8}{L}$. There, real solutions have been specified by non-repeating indices $\{\ell_i | i = 1, 2, \dots, N\}$ and m. But, for $0 < U < \frac{8}{L}$, real solutions with repeating indices may appear at the same time as k-Λ-2-string solutions disappear. Such redistribution of type of solutions is observed in the isotropic Heisenberg model for a large number of sites [4, 8, 9]. Here we numerically investigate such phenomenon for the Hubbard model with $L = 20$ and $N = 2$ [3].

As we have already mentioned, k-Λ-2-string solutions with odd m may disappear for $0 < U < \frac{8}{L}$ (see Figure 3). For each m, the critical value of U is exactly given by [3],

$$U^{(m)} = -\frac{8}{L}\cos\left(\frac{\pi}{L}m\right) < \frac{8}{L}.$$

Plotted on Figure 4 are the center $\zeta = \frac{\pi}{L}m, (m = 11, 12, \dots, 29)$ of k-Λ-2-string solutions and redistributed real solutions $\{k_1, k_2\}$ for $L = 20$ and $N = 2$ in varying the value of U. Plotted

on Figure 5 are their imaginary parts. The k-Λ-2-string solutions with odd m disappear for $U < U^{(m)}$ and, at the same time, real solutions with repeating indices $\{\ell_1, \ell_2; m\} = \{\frac{m}{2}, \frac{m}{2}; m\}$ appear. Note that, as $U \to 0$, all the k-Λ-2-string solutions on Figure 4 approach to the wavenumbers of lattice free fermion system. Thus, when we apply the combinatorial formulas in Corollary 4.7 to the case, we must count the number of Bethe states by regarding the real solutions with repeating indices as k-Λ-string solutions.

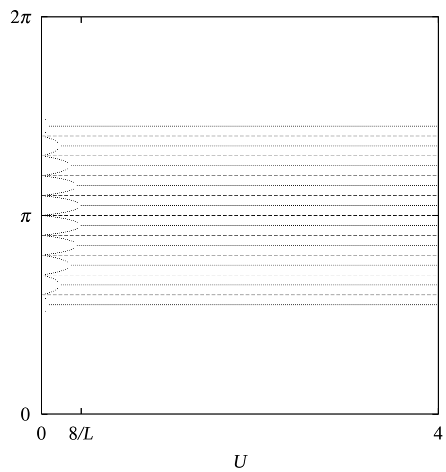

Figure 4: The real parts of solutions for Lieb-Wu equations with $L = 20$ and $N = 2$. The dashed lines correspond to the centers ζ of k-Λ-2-string solutions with even m, and the dots express the centers ζ of k-Λ-2-string solutions with odd m and their redistribute real solutions.

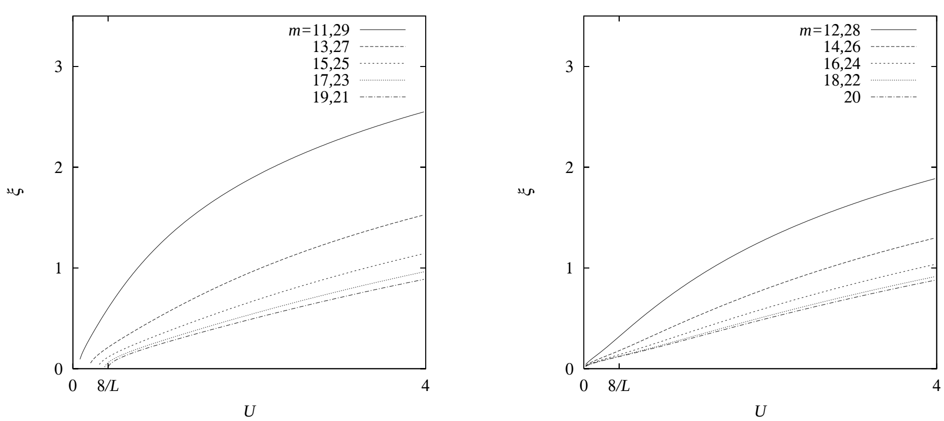

Figure 5: The imaginary parts ξ of solutions for Lieb-Wu equations with $L = 20$ and $N = 2$.

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